Injective Continuous Reduction on the Borel subsets of the Baire space.

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section Set Theory & Topology

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What is a reduction?

- Fix a topological space X;
- Fix a reduction condition (i.e. a class of function $\mathcal{F}$ from and to X such that id_X is in $\mathcal{F}$ and $\mathcal{F}$ is closed under composition).

Let $A, B \in \mathbf{\Gamma} \subseteq \mathcal{P}(X)$

$$A \leqslant B \Leftrightarrow \exists f \in \mathcal{F} \ (f^{-1}(B) = A)$$

That is a classification of subsets in $\mathbf{\Gamma}$ through $\mathcal{F}$.
The structure $(\mathbf{\Gamma}, \leqslant)$ is a quasi-order, that is reflexive and transitive.

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Examples:

- $X = \mathbb{N}^{\mathbb{N}}$, $\mathbf{\Gamma} = \mathrm{BOR}(\mathbb{N}^{\mathbb{N}})$
  - Wadge Reduction (Wadge, 1972) $\mathcal{F} = \{f \mid f \text{ is continuous}\}$;
  - Borel Reduction (Andretta and Martin, 2003) $\mathcal{F} = \{f \mid f \text{ is Borel}\}$;
  - Borel-amenable Reduction (Motto Ros, 2007) $\mathcal{F}$ is amenable (for example $\mathbf{\Delta}^0_\xi$-function);
  - Contraction Reduction (Motto Ros, 2012) $\mathcal{F} = \{f \mid f \text{ is Lipschitz with a positive constant } < 1\}$;
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- Wadge Reduction on $\mathbb{R}$ (Hertling 1996, Schlicht 2012) $X = \mathbb{R}$, $\mathbf{\Gamma} = \mathrm{BOR}(\mathbb{R})$, $\mathcal{F} = \{f \mid f \text{ is continous}\}$.

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Fix $X, \mathbf{\Gamma}$ and $\mathcal{F}$, we can define a partial order induced by $\mathcal{F}$, i.e. $(\mathbf{\Gamma}, \leqslant)\big/_{\equiv}$, where $A \equiv B$ if and only if $A \leqslant B$ and $B \leqslant A$.

First Goal: Description of the partial order induced by the reduction.

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What does "description" mean in this talk?
It means answering the question: Is it wqo?

Definition

Let $(Q, \leqslant)$ be a partial order, then Q is wqo if

- Q is well-founded, i.e. there are no infinite strictly decreasing sequence;
- there are no infinite antichains, i.e. any infinite $A \subseteq Q$ admits p and q such that $p \leqslant q$.

Examples of wqo: $(\mathbb{N}, \leqslant)$, $(\alpha, \leqslant)$, $(\mathbb{N}^n, \leqslant_{\text{prod-ord}})$.
Examples of not wqo: $(\mathbb{N}, |)$, $(\mathbb{N}^{\mathbb{N}}, \leqslant_{\text{lex}})$, $(\mathbb{N}, =)$.

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Examples:

- Wadge Reduction (Wadge, 1972) Good
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The Baire space $\mathbb{N}^{\mathbb{N}}$ is a zero-dimensional Polish space.
Its topology is generated by $\mathrm{N}_s = \{x \in \mathbb{N}^{\mathbb{N}} \mid s \sqsubseteq x\}$, where $s \in \mathbb{N}^{<\mathbb{N}}$.

$$\begin{array}{ccc} \boldsymbol{\Sigma}^0_1 & \boldsymbol{\Pi}^0_1 & \boldsymbol{\Delta}^0_1 \\ \boldsymbol{\Sigma}^0_2 & \boldsymbol{\Pi}^0_2 & \boldsymbol{\Delta}^0_2 \end{array}$$

From now assume that $\boldsymbol{\Gamma}$ is closed under continuous preimage, that is if $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is continuous and $A \in \boldsymbol{\Gamma}$ then $f^{-1}(A) \in \boldsymbol{\Gamma}$.

Examples:

- $\boldsymbol{\Sigma}^0_2$
- $\mathrm{Diff}_3(\boldsymbol{\Sigma}^0_1(\mathbb{N}^{\mathbb{N}}))$

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What about the description of the Wadge reduction in the class of zero-dimensional Polish spaces?

Theorem (R. Carroy, L. Motto Ros, S. - 202?)

For each zero-dimensional Polish space, there exists a pair of ordinals which completely determines the structure of $\mathcal{W}_X$ up to order-isomorphism.

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Let X be a zero-dimensional Polish space with at least two points, and assume AD if X is uncountable. Then there is no Borel procedure to determine which zero-dimensional Polish spaces Y gives $\mathcal{W}_Y \equiv_{\mathrm{W}} \mathcal{W}_X$.

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Class of function:

$$\mathcal{F}_i = \{f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} \mid f \text{ is continuous and injective}\}$$

Definition

Let $A, B \subseteq \mathbb{N}^{\mathbb{N}}$, we write $A \leqslant_{\mathrm{i}} B$ if and only if there exists $f \in \mathcal{F}_i$ such that $f^{-1}(B) = A$.

Corollary of Theorem (F. van Engelen, A. W. Miller, J. R. Steel - 1985)

The partial order $(\mathbf{\Delta}^0_2, \leqslant_{\mathrm{i}})$ is wqo.

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Definition: a subset $A \subseteq 2^{\mathbb{N}}$ is true in $\boldsymbol{\Gamma}$ if $A \in \boldsymbol{\Gamma}$ then $2^{\mathbb{N}} \setminus A \notin \boldsymbol{\Gamma}$.

Definition: $\boldsymbol{\Gamma} \subseteq \mathrm{BOR}(2^{\mathbb{N}})$ is reasonably closed if ...

Proposition (L. Harrington, ? - J. R. Steel, 1977)

Let $\boldsymbol{\Gamma} \subseteq \mathrm{BOR}(2^{\mathbb{N}})$ such that $\boldsymbol{\Gamma}$ is reasonably closed and $A, B \in \boldsymbol{\Gamma}$. If B is a true set for $\boldsymbol{\Gamma}$ and $A \leqslant_{\mathrm{W}} B$ then $A \leqslant_{\mathrm{i}} B$.

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Let $\boldsymbol{\Gamma} \subseteq \mathrm{BOR}(2^{\mathbb{N}})$ such that $\check{\boldsymbol{\Gamma}} \neq \boldsymbol{\Gamma}$. If $\mathcal{L}(\boldsymbol{\Gamma}) \geqslant 1$ and $\mathrm{Diff}_n(\boldsymbol{\Sigma}^0_2(2^{\mathbb{N}})) \subseteq \boldsymbol{\Gamma}$ for each $n \in \omega$ then $\boldsymbol{\Gamma}$ is reasonably closed.

There are no trivial classes $\boldsymbol{\Gamma}$ that are not reasonably closed, for example $\mathrm{Diff}_2(\boldsymbol{\Sigma}^0_2(2^{\mathbb{N}}))$.

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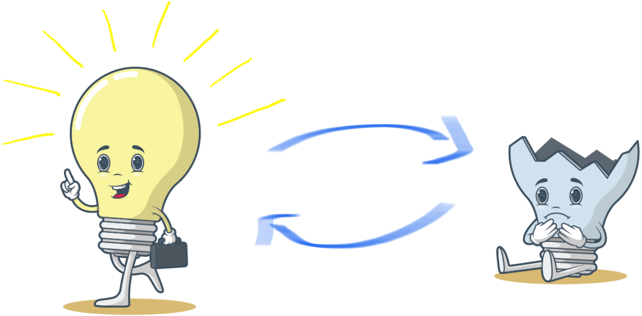
Work in progress...



Definition: $\mathcal{F}_i^{\mathbf{\Pi^0_1}} = \{f \in \mathcal{F}_i \mid \operatorname{ran} f \text{ is closed}\}$. Let us notice $\leqslant_{\mathrm{i}}^{\mathbf{\Pi}^0_1}$ its induced reduction.

Conjecture: $((\mathbf{\Pi^0_1})_{\aleph_0}, \leqslant_{\mathrm{i}})$ is a linear order.

Theorem (R. Carroy, L. Motto Ros, S.)

The partial order $((\mathbf{\Pi^0_1})_{\aleph_0}, \leqslant_{\mathrm{i}}^{\mathbf{\Pi}^0_1})$ admits an antichain of size n for each $n \in \omega$.

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